Received: *March 23, 1999*, Accepted: *June 2, 1999*

Geometrical aspects of fivebranes in heterotic/F-Theory duality in four dimensions

Duiliu-Emanuel Diaconescu and Govindan Rajesh

School of Natural Sciences, Institute for Advanced Study,
Olden Lane, Princeton, NJ 08540
E-mail: diacones@ias.edu, rajesh@ias.edu

Abstract: We use the method of stable degenerations to study the local geometry of Calabi–Yau fourfolds for F-theory compactifications dual to heterotic compactifications on a Calabi–Yau threefold with fivebranes wrapping holomorphic curves in the threefold. When fivebranes wrap intersecting curves, or when many fivebranes wrap the same curve, the dual fourfolds degenerate in interesting ways. We find that some of these can be usefully described in terms of degenerations of the base of the elliptic fibrations of these fourfolds. We use Witten's criterion to determine which of the fivebranes can lead to the generation of a non-perturbative superpotential.

Keywords: String Duality, Superstrings and Heterotic Strings, F-Theory.

JHEP06(1999)002

Contents

1. Introduction

Heterotic compactifications on elliptic Calabi–Yau threefolds provide us with phenomenologically interesting vacua in four dimensions with $N = 1$ supersymmetry. Moreover, such vacua are conjectured to have a dual description in terms of F-Theory compactified on elliptically fibered Calabi–Yau fourfolds. The heterotic vacua are specified by a choice of gauge bundle, constructed using the general techniques of Friedman, Morgan and Witten [1]. It was shown there that such vacua often include fivebranes wrapping the elliptic fibers of the Calabi–Yau threefold Z, whose number is determined by an anomaly cancelation condition.

More generally, we can have fivebranes on the heterotic side wrapping holomorphic curves in Z. In this paper, we examine the geometry of fourfold duals of heterotic vacua with fivebranes. The cohomology classes of the curves being wrapped are fixed by the general heterotic anomaly cancelation condition

$$[W] = c_2(TZ) - \lambda(V_1) - \lambda(V_2)\,, \tag{1.1}$$

where $[W]$ is the class of the wrapped curves, $c_2(TZ)$ is the second Chern class of the tangent bundle of the Calabi–Yau threefold Z, and $\lambda(V_i)$ are the second Chern classes of the vector bundles on Z. Note that this only fixes the curves up to their cohomology classes. Given that Z is elliptically fibered with a section σ, we see that the class $[W]$ may be written $[W] = C_1\sigma + C_2$, so that under the projection $\pi : Z \to B_2$, C_1 maps to a divisor in B_2, and C_2 maps to $h[p]$, where $[p]$ is the class of a point in B_2 and $h = \int_\sigma C_2$ is an integer. Thus the class C_2 actually describes the fivebranes wrapping the elliptic fiber of Z, while $C_1\sigma$ describes the fivebranes wrapping holomorphic curves in the base B_2. In this paper, we will refer to the first kind (i.e. those in the class C_2) as vertical fivebranes, and the second (i.e. those in the class $C_1\sigma$) as horizontal ones. (It is of course possible for a fivebrane to wrap a curve which has both vertical (i.e. fiber) components and horizontal (i.e. base) components.)

Under the duality, the vertical fivebranes map to F-theory threebranes [2]. The number of the F-theory threebranes is related to the Euler number of the fourfold by tadpole anomaly cancelation [3]. This relation is actually modified in the presence of the four form field strength G. Also, when the threebranes coincide with the sevenbranes wrapping divisors in the base B_3 of the fourfold over which the elliptic fibration degenerates, they behave as instantons, breaking the gauge group observed from the singularity to a smaller one. For the purposes of this paper, we will ignore both these possibilities, since they play no role in our analysis. Our results will be valid even in the presence of these complications.

The horizontal fivebranes, on the other hand, map to geometric data on the F-theory side. Specifically, if a fivebrane wraps a curve C in B_2, then the F-theory base B_3 is blown up once over the corresponding curve [4] fivebranes wrapping the same curve correspond to an equal number of blow-ups on the F-theory side.

For the purposes of this paper, we choose to ignore the vector bundles on the heterotic theory altogether, and concentrate instead on the local physics of the fivebranes. Thus, we consider an extreme situation when the bundles $V_{1,2}$ are without structure group. This is analogous to the six dimensional vacuum with 24 small instantons, the anomaly cancelation being entirely due to fivebranes. The base of the F-Theory threefold in this case acquires several blowups [5], whose local description involves the method of stable degenerations [6, 7].

In the four dimensional situation, it is precisely the horizontal fivebranes that correspond to blowup modes in the fourfold base. In this paper, will use the method of stable degenerations to describe this geometry. After a brief description of the general technique in section 2, we explicitly work out the fourfold geometry in the stable degeneration limit for a single horizontal fivebrane in section 3. When two fivebranes wrap intersecting curves, or when several fivebranes wrap the same curve, the dual fourfolds degenerate in interesting ways. We find that it is more useful to describe these degenerations by studying the degenerations of the corresponding base B_3, which is the subject of section 4. We find, for example, that when two horizontal fivebranes intersect, the base B_3 acquires a conifold singularity, while k horizontal fivebranes wrapping the same curve C in B_2 lead to an A_{k-1} singularity fibered over the corresponding curve in B_3.

JHEP06(1999)002

In addition to affecting the geometry of the fourfolds, the fivebranes can contribute to the nonperturbative superpotential. Section 5 is devoted to discussing the criteria for determining which fivebranes can contribute to the superpotential, based on the work of [8, 9].

2. Heterotic models and stable degenerations

This section consists of a brief review of heterotic vacua and the stable degeneration limit. Consider a general $d = 4$, $N = 1$ heterotic vacuum specified by the compactification data (Z, V_1, V_2). Here $\pi : Z \to B_2$ is a nonsingular elliptic Calabi-Yau threefold with a section $\sigma : B_2 \to Z$. V_1, V_2 are two holomorphic bundles with structure group G_1^c, G_2^c where $G_{1,2} \subset E_8$. Throughout the paper, the base B_2 will be taken to be a Hirzebruch surface F_e, with $e = 0, 1, 2$ in order to insure the smoothness of the total space Z. Moreover, the present considerations are restricted to heterotic models which admit an F-theory dual. Therefore, Z will be taken to be a smooth Weierstrass model

$$zy^2 = x^3 - axz^2 - bz^3 \tag{2.1}$$

in $\mathbf{P}\left(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3\right)$ with $\mathcal{L} \simeq K_{B_2}^{-1}$ in order to satisfy the Calabi-Yau condition. a, b are sections of $\mathcal{L}^4, \mathcal{L}^6$. The bundles $V_{1,2}$ will be specified by spectral data $(\Sigma_{1,2}, \mathcal{N}_{1,2})$ [1, 10, 11].

According to [5, 1], the dual F-theory model can be constructed by taking the size of the base B_2 very large so that we can use adiabatic arguments. The elliptic fibration $\pi : Z \to B_2$ is then replaced adiabatically by a $K3$ fibration over B_2 with total space a Calabi-Yau fourfold X. It turns out that X can be represented as an elliptic Weierstrass model $\pi' : X \to B_3$ where $p : B_3 \to B_2$ is a rationally ruled threefold over B_2. The moduli map between the heterotic and F-theory data has been discussed intensively in [5, 1, 10, 12, 4, 13].

A particularly convenient approach is the method of stable degenerations [1, 7, 14] which establishes a direct geometric correspondence between the two sets of data. Briefly, this consists of taking a limit in which the size of the elliptic fiber of Z is also very large. Then, the F-theory fourfold X degenerates to a union of two fourfolds $X = X_1 \sqcup_Z X_2$ glued together along a three dimensional variety isomorphic to Z. Both X_1, X_2 are elliptically fibered over rationally ruled threefolds isomorphic to B_3 with projections π_1', π_2' such that $\pi = \pi_1'|_Z = \pi_2'|_Z$. The composite maps $p \circ \pi_{1,2}'$ are fibrations of $X_{1,2}$ over B_2, with generic fiber a rational elliptic surface dP_9.

As explained in [7, 14] this procedure is very useful for studying the F-theory local geometry associated to heterotic small instantons. However, since the stable degeneration is, strictly speaking, at infinite distance in the moduli space metric, it may not be suitable for describing particular physical processes. This will be the case with certain fivebrane interactions and nonperturbative instanton effects.

JHEP06(1999)002

These phenomena are better described using a smooth resolution of the Calabi-Yau fourfold in a region of the moduli space where the geometric map developed via stable degenerations is still valid.

3. A single horizontal fivebrane

Here we consider the case of a single heterotic fivebrane wrapping a nonsingular irreducible curve $C \subset B_2$. The goal of this section is to determine the corresponding F-theory geometry in the stable degeneration limit. Recall that the fourfold $X \equiv X_1$ can be represented as a Weierstrass model over a base B_3 which can be identified with the total space of a projective bundle $\mathbf{P}\left(\mathcal{O}_{B_2} \oplus \mathcal{T}\right)$ over B. In the following, we will restrict to models in which $\mathcal{T} \simeq \mathcal{O}_{B_2}(-\Gamma)$ for some effective divisor Γ on B_2. To introduce some notation, note that the fibration $p : B_3 \to B_2$ has two disjoint holomorphic sections which will be denoted by S_0, S_∞ by analogy with Hirzebruch surfaces. Then

$$\begin{aligned} S_\infty &= S_0 + p^*\Gamma \\ K_{B_3} &= -2S_0 + p^*\left(K_{B_2} - \Gamma\right). \end{aligned} \tag{3.1}$$

The fourfold $\pi' : X \to B_3$ is described as a Weierstrass model

$$y^2 = x^3 - fx - g \tag{3.2}$$

in $\mathbf{P}\left(\mathcal{O}_{B_3} \oplus \mathcal{L}'^2 \oplus \mathcal{L}'^3\right)$, with $\mathcal{L}' = \mathcal{O}_{B_3}(S_0) \otimes p^* K_{B_2}^{-1}$. Note that the total space X is not a Calabi-Yau variety in this case. As in [7, 6], the structure of the elliptic fibrations is determined by the divisors

$$\begin{aligned} F &= 4S_0 - 4p^*K_{B_2} \\ G &= 6S_0 - 6p^*K_{B_2} \\ \Delta &= 12S_0 - 12p^*K_{B_2}. \end{aligned} \tag{3.3}$$

The base B_3 is glued to the base of the second fourfold of the stable degeneration along the section S_∞. Therefore the gluing divisor is the restriction of the elliptic fibration to S_∞ which can be described as a Weierstrass model with line bundle $K_{B_2}^{-1}$. This generically defines a smooth Calabi-Yau space which is identified with Z. Also, note that the restriction of the elliptic fibration to the generic rational fiber of $p : B_3 \to B_2$ has exactly 12 I_1 fibers, therefore it is a rational elliptic surface as claimed before.

Since we are interested in heterotic models without structure group, we first enforce a section of II^* singularities along the section S_0 of B_3 ensuring an unbroken E_8 group. This corresponds to splitting the divisors F, G, Δ into

$$\begin{aligned} F &= 4S_0 + F', & F' &= -4p^*K_{B_2} \\ G &= 5S_0 + G', & G' &= S_0 - 6p^*K_{B_2} \\ \Delta &= 10S_0 + \Delta', & \Delta' &= 2S_0 - 12p^*K_{B_2}. \end{aligned} \tag{3.4}$$

The location of the small instantons will be determined by the collision of the component Δ' of the discriminant with the section S_0. A more precise description requires the explicit expression of the polynomials f, g, δ function of the coordinates of the base. In fact, the Weierstrass model can be characterized in a neighborhood of $S_0 \simeq B_2$ as a hypersurface

$$y^2 = x^3 - fx - g \tag{3.5}$$

in the bundle $\mathcal{T} \oplus \mathcal{T}^2 \otimes \mathcal{L}^2 \oplus \mathcal{T}^3 \oplus \mathcal{L}^3$ [1]. Let s denote an affine coordinate along the fibers of B_3 in the neighborhood of S_0. Hence s is a section of the normal bundle $N_{S_0/B_3} \simeq \mathcal{T}$. Then we have the following power series expansions

$$\begin{aligned} f &= s^4 f_4 \equiv s^4 f' \\ g &= s^5 \left(g_5 + s g_6\right) \equiv s^5 g' \\ \delta &= s^{10} \left[4 s^2 f_4^3 + 27 \left(g_5 + s g_6\right)^2\right] \equiv s^{10} \delta' , \end{aligned} \tag{3.6}$$

where f_4 is a section of $\mathcal{L}^4$ and g_5, g_6 are sections of $\mathcal{L}^6 \otimes \mathcal{T}, \mathcal{L}^6$.

The component Δ' is defined by $\delta' = 0$. This is a quadratic equation in s with discriminant (up to a multiplicative factor)

$$f_4^3 g_5^2 \, . \tag{3.7}$$

Therefore Δ' is a double cover of B_2 branched along the locus

$$f_4 = g_5 = 0 \, . \tag{3.8}$$

For future reference, let A, C denote the loci $f_4 = 0$ and $g_5 = 0$ in $S_0 \simeq B_2$ respectively. By a suitable choice of the line bundle $\mathcal{T}$, these can be assumed nonsingular irreducible curves. Note that C actually represents the locus of intersection of Δ' with S_0, which is precisely the location of the small instanton. The singularity of the total space X induced by the $I_1 + II^*$ collision along C requires a blow-up of the base, as will be shortly detailed. The points of the discriminant in the inverse image of A in Δ' represent a locus of II elliptic fibers of X. Furthermore, note that Δ' intersects the section S_∞ along the locus L given by

$$4 f_4^3 + 27 g_6^2 = 0 \, , \tag{3.9}$$

which is a section of $-12K_{B_2}$. The equation $\delta' = 0$ is identically satisfied when

$$4 f_4^3 + 27 g_6^2 = 0 \, , \qquad g_5 = 0 \tag{3.10}$$

hold simultaneously. Therefore, the discriminant δ' contains the $-12K_{B_2} \cdot C$ fibers of $p : B_3 \to B_2$ localized at the points of intersection $C \cdot L$ in B_2.

As observed above, a smooth model of the fourfold X can be constructed by blowing-up the base B_3 along the curve C embedded in S_0. In order to describe the resulting configuration, let (t, u) be local coordinates on S_0 near a point P of C so that t is a normal coordinate and u is a coordinate on C centered at P. The section g_5 is assumed to have a simple zero along C, therefore, it will have a local expansion of the form

$$g_5 = t g_5' \,, \tag{3.11}$$

where g_5' does not vanish at $t = 0$. The blow-up is described in affine coordinates by setting $s = s_1 t_1, t = t_1$ which results in

$$\begin{aligned} f &= s_1^4 t_1^4 f_4 \\ g &= s_1^5 t_1^6 (g_5' + s_1 g_6) \\ \delta &= s_1^{10} t_1^{12} \left[4 s_1^2 f_4^3 + 27 \left(g_5' + s_1 g_6 \right)^2 \right] . \end{aligned} \tag{3.12}$$

The Weierstrass model can be set in normal form by rescaling the coordinates x, y by appropriate powers of t_1 obtaining

$$\begin{aligned} f &= s_1^4 f_4 \\ g &= s_1^5 (g_5 + s_1 t_1 g_6) \\ \delta &= s_1^{10} \left[4 s_1^2 f_4^3 + 27 \left(g_5' + s_1 g_6 \right)^2 \right] . \end{aligned} \tag{3.13}$$

Globally, the exceptional divisor D is isomorphic to the projective bundle $\mathbf{P}\left(N_{C/B_3}\right)$ whose P^1 fiber is parameterized by the affine coordinate s_1. The degree of the ruling is $-6K_{B_2} \cdot C$. The strict transform of the vertical divisor p^*C is isomorphic to another rationally ruled surface D' over C. Let $C_0^D, C_\infty^D, C_0^{D'}, C_\infty^{D'}$ denote the disjoint sections of D, D' respectively and let S_0' denote the strict transform of the section C_0. Then S_0' and D intersect along C_0^D, and D and D' intersect along a common section $C_0^D \simeq C_\infty^{D'}$. The blown-up base $\bar{B}_3$ can be regarded as a fibration $\bar{p} : \bar{B}_3 \to B_2$ with generic fiber P^1. Above $C \subset B_2$, the fiber consists of two rational components intersecting transversely. See fig. 1. for a schematic representation.

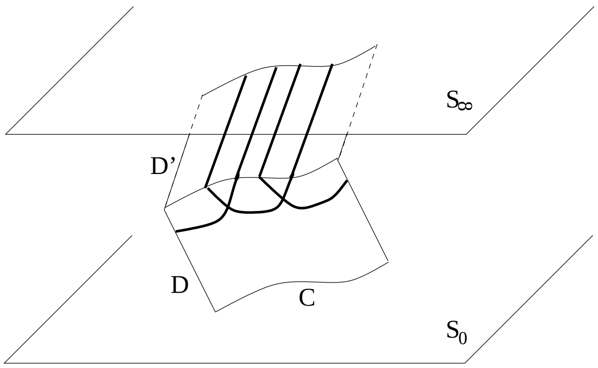

Figure 1: The blow-up geometry corresponding to a single horizontal fivebrane wrapped around the curve C. D and D' are the exceptional divisor and the proper transform of p^*C respectively. The thick lines represent the intersection of the discriminant with D, D'.

It follows from (3.13) that the proper transform of the discriminant $\bar{\Delta}'$ intersects D along a divisor in the class $2C_\infty^D$ which does not meet S_0'. Therefore S_0' is an isolated section of II^* fibers of the resulting fourfold $\bar{X}$. Furthermore, $\bar{\Delta}'$ intersects the surface D' along $-12K_{B_2} \cdot C$ P^1 fibers which carry I_1 elliptic fibers. An interesting exception arises in the case $B_2 \simeq F_2$ and $C = C_0$. In this case, the exceptional divisor D is simply $P^1 \times P^1$ since $K_{B_2} \cdot C_0 = 0$ and $\bar{\Delta}'$ intersects D along two disjoint sections. The strict transform D' does not intersect $\bar{\Delta}'$ and there are no vertical lines carrying I_1 elliptic fibers.

If everything else is generic, the elliptic fibration is smooth away from S_0' and it has a locus of type II fibers which projects to the curve A in the base. From here, a smooth model is easily obtained by blowing up $\bar{X}$ along the section S_0', obtaining an E_8 Hirzebruch-Jung tree fibered over S_0'. These exceptional divisors play an important role in the dynamics of $d = 3$ $N = 2$ pure gauge theories [15, 16].

3.1 The spectral cover

The heterotic spectral cover corresponding to the above fourfold degeneration can be determined by analogy with the six dimensional situation considered in [7, 14]. In that case, it is known [5], that the extra Kähler moduli associated to a blow-up in the base corresponds to an extra $(1,0)$ tensor multiplet. This allows an identification of the threefold degeneration with a small E_8 instanton, that is a heterotic fivebrane. In the present situation, the low energy effective action corresponding to the fourfold degeneration can be easily derived regarding the model as a IIB compactification on the base $\bar{B}_3$ with a varying dilaton. Note that blowing-up B_3 along the curve C has the effect of producing nontrivial homology 3-cycles. More precisely, the intermediate jacobian $J\left(\bar{B}_3\right)$ is isomorphic to the jacobian $J\left(C\right)$ [17, 18]. Therefore there is a $1-1$ correspondence between $H^{1,2}\left(\bar{B}_3\right)$ and $H^{0,1}\left(C\right)$ given by the cylinder (or equivalently, Abel-Jacobi) map of [17, 18]. Note that $h^{0,3}\left(\bar{B}_3\right) = h^{0,3}(B_3) = 0$, therefore $J\left(\bar{B}_3\right)$ is in this case a principally polarized abelian variety.

The low energy effective action is then determined by reducing the $\mathrm{SL}(2,\mathbb{Z})$ invariant 4-form $C^{(4)}$ with self-dual field strength $G^{(5)}$ along the elements of $H^{1,2}\left(\bar{B}_3\right)$. The discussion is similar to the reduction of the M-theory fivebrane along a compact Riemann surface found in [8]. We have an ansatz

$$G^{(5)} = F \wedge \Lambda + *F \wedge *\Lambda\,, \tag{3.14}$$

where F is a 2-form on R^4 and Λ is a 3-form on $\bar{B}_3$. The equation of motion $dG^{(5)} = 0$ yields the Maxwell equations for F and requires Λ to be a harmonic 3-form. It follows from Hodge theorem that Λ defines a point in the intermediate jacobian $J\left(\bar{B}_3\right)$. The corresponding low-energy effective action consists of $\mathrm{U}(1)^g$ massless gauge fields whose couplings are determined by $J\left(\bar{B}_3\right)$ (g is the genus of the wrapped Riemann surface). Taking into account the identification $J\left(\bar{B}_3\right) \simeq J\left(C\right)$, we conclude that the effective action derived this way is in fact identical with that of an M-theory fivebrane wrapping the compact Riemann surface C [19, 20]. This provides physical evidence for identifying the fourfold degeneration with an E_8 heterotic fivebrane wrapping the curve C in the base B_2.

JHEP06(1999)002

A more precise geometric picture can be achieved as follows. Note that the strict transform D' discussed in the previous subsection intersects the section S_∞ along a curve isomorphic to C. The restriction of the elliptic fibration to D' gives an elliptic threefold $Q \to D'$ with I_1 degenerations along $-12K_{B_2} \cdot C$ rational fibers of D'. Moreover, the heterotic Calabi-Yau threefold Z is identified with the restriction of $\bar{\pi}' : \bar{X} \to \bar{B}_3$ to S_∞. Therefore Q and Z meet along a surface S elliptically fibered over C and with $-12K_{B_2} \cdot C$ I_1 fibers. As in [14], this is an irreducible component of the spectral cover $\Sigma = \sigma \cup S$. Note that the reducible spectral cover describes a "bundle without structure group" which is in proper language the ideal sheaf $\mathcal{J}_{C/Z}$. Similar degenerations have been also considered in [21].

Before further developing the subject by considering multiple small instantons, a couple of remarks are in order. Some aspects of the present construction may be better understood by comparison with the threefold degenerations studied in section 3.4 of [14]. Note that in that case, the role of the elliptic threefold Q is played by an elliptic surface which is generically trivial i.e. $Q \simeq P^1 \times T^2$. The injective map $H^1(Q, Z) \to H^3(X, Z)$ induces nontrivial homology 3-cycles on X. The spectral cover is again reducible $\Sigma = \sigma \cup S$ where S is isomorphic to the constant elliptic fiber of Q. Therefore, the jacobian of Σ, which is isomorphic to that of S, is mapped injectively into the intermediate jacobian of X. This is part of the Heterotic/F-theory map which maps the position of the small instanton along the elliptic fiber to Ramond-Ramond moduli of X. In the present situation, S is an elliptic surface which can be written as a Weierstrass model over $C \subset \sigma$ with line bundle $K_{B_2}^{-1}|_C$. This shows that the normal bundle $N_{C/S} \simeq K_{B_2}|_C$ is of negative degree, therefore C cannot be deformed in S. Accordingly, there are no moduli parameterizing the position of the small instanton along the elliptic fiber. If the genus of C is $g \geq 1$, the surface S has a nontrivial jacobian isomorphic to that of C which injects in the jacobian of X as explained before. This is again part of the Heterotic/F-theory map having, however, a different physical interpretation in terms of the effective couplings of the four-dimensional low-energy action.

Note that there is an exception to this generic behavior. Namely, as also noted before, if $B \simeq F_2$ and $C = C_0$, it turns out that the threefold Q and the surface $\Sigma = Q \cap Z$ are trivial elliptic fibrations, that is $Q \simeq D' \times T^2$ and $\Sigma \simeq P^1 \times T^2$. This behavior is very similar to the six dimensional case since the jacobian of Σ is now isomorphic to that of the trivial elliptic fiber. In particular, the fivebrane wrapped on $C \subset \sigma$ can be moved along the elliptic fiber and its position on T^2 is parameterized by a point in the jacobian.

3.2 Interaction with vertical fivebranes — a puzzle

We know from anomaly cancelation considerations that vertical fivebranes should also be present. Since they are obviously mobile along the heterotic base, the vertical fivebranes can collide the horizontal brane by a suitable tuning of moduli. For simplicity, we consider a single vertical fivebrane approaching the horizontal one. The effective theory on the non-compact directions of the vertical fivebrane is a free U(1) gauge theory with three neutral complex chiral multiplets. Two chiral multiplets parameterize the motion of the fivebrane along the base. The scalar component of the third, $\Phi = \phi + ia$, incorporates the position ϕ along the interval $S^1/\mathbb{Z}_2$ and the fivebrane axion $da = *_4 dB$.

In flat eleven dimensional space, such a collision is expected to result in extra degrees of freedom — "tensionless strings" — localized on the intersection and eventually in an interacting superconformal fixed point [22, 23]. In our case, the fivebranes wrap Riemann surfaces embedded in a curved space, therefore there could be extra effects and it is not clear if the decoupling takes place.

A possible approach to this problem is via duality with F-theory. As noted above, the horizontal fivebranes map to background threebranes filling the non-compact $3+1$ directions. The expectation value of the field Φ is related to the position of the threebrane on the P^1 fiber of $p : B_3 \to B$. In particular if the vertical fivebrane is localized at a point $P \in C$ on the base B_2, Φ corresponds to the threebrane position along the P^1 fiber of the proper transform D'. Therefore, the collision takes place precisely when the threebrane hits the exceptional divisor D. However, this apparently leads to a puzzle since the total space of the threefold B_3 is smooth hence, we have no reasons to expect an interacting theory on the threebrane worldvolume when it collides with the intersection $D' \cap D$. In particular, although the heterotic picture suggests a solitonic string of vanishing tension localized on the intersection of the fivebranes, no such object can be found in the threebrane worldvolume theory. At the present stage, although we do not have a complete solution of this puzzle, we suggest that it could be understood along the following lines. The present picture is valid within the framework of adiabatic duality, *i.e.*, the size of the base B_2 is much larger than the size of the elliptic fiber. Note that this is not the case in the stable degeneration limit, therefore the analysis is performed using a smooth F-theory model in a suitable region of the moduli space so that the local geometry described by the stable degeneration is still valid. Then, the intersecting fivebranes can be locally modeled as two intersecting fivebranes in M-theory on $R^{1,8} \times T^2$. The vertical fivebrane corresponds to a fivebrane whose relative transverse directions are wrapped on T^2, while the horizontal fivebrane is transverse to the torus. Using standard duality arguments, this configuration can be mapped to a D3-brane in Type IIB theory moving in a smooth Taub-NUT background. Therefore the theory of the brane has a smooth moduli space, similarly to the situation considered in [24]. In particular, there are no singularities and no extra light degrees of freedom at any point in the moduli space. This is in agreement with the F-theory picture developed above.

Another question is related to the possibility of deforming the intersecting fivebranes, obtaining a single brane wrapping a smooth irreducible curve in the Calabi-Yau threefold. This is an interesting question also treated in [20]. For concreteness, let us consider a fivebrane wrapping an effective cycle in the homology class $C+nf$ where f is the class of the elliptic fiber. As shown in a particular example in [20], the moduli space of this fivebrane has, in general, many disconnected components. More precisely, let $\Sigma = \pi^*C$ and $i : \Sigma \to Z$ denote the inclusion. There could several homologically inequivalent configurations of holomorphic curves on Σ mapping to the homology class $C+nf \in H_2(Z)$ under the map $i_* : H_2(\Sigma) \to H_2(Z)$. However, although these curves are homologically equivalent when embedded in Z, they may not be algebraically equivalent, in which case the different types of fivebranes cannot be deformed one to another in a supersymmetric way. This gives rise to several distinct components of the fivebrane moduli space.

For the class $C+nf$, one can show that there are no algebraic (i.e. holomorphic) deformations that lie entirely within $\Sigma = \pi^*C$. However, it may be possible to deform this class in the ambient space Z. Such deformations, cannot, however, exist if C has no deformations in B_2 (e.g. if $C \cdot C < 0$). This is because any deformation of $C+nf$ into a smooth irreducible curve C' in Z must project down to a deformation of C in B_2, but C cannot move in B_2, which implies that the deformation lies entirely in Σ, and hence does not exist. Thus, if there is a curve C' in Σ which is different from $C+nf$ in $H_2(\Sigma)$, but maps to the same element of $H_2(Z)$ under i_*, then we cannot deform holomorphically from $C+nf$ to C', so that they sit in disconnected components of the fivebrane moduli space. However, it is not clear if such deformations exist if C moves in B_2. Deformations entirely within Σ can exist when the multiplicity of the horizontal fivebranes is greater than one, i.e. the fivebrane class is $kC+nf$, with $k>1$. This phenomenon will be considered at a later stage.

4. Multiple fivebranes

4.1 Intersecting horizontal fivebranes

The discussion in the previous section can be generalized in several different ways. We consider here the situation when we have two horizontal fivebranes wrapping two irreducible smooth curves $C_1, C_2 \subset B_2$ intersecting in a finite number of points. These curves may belong to the same, or different cohomology class in the base. In [4], the dual F-theory picture was described in terms of intersecting del Pezzo surfaces. Here, we will describe it in terms of the geometry of the fourfold base. This corresponds to viewing F-theory as a nonperturbative Type IIB vacuum, which, as we will see shortly, is a more useful picture.

For simplicity, we first consider the case when both C_1, C_2 are rational and they intersect in exactly one point (for example a fiber and a section of a Hirzebruch surface). The base of the fourfold $\bar{B}_3$ is now given by the P^1 bundle $p : B_3 \to B$ blown-up twice along the curves C_1, C_2 embedded in the section S_0. Since the curves are intersecting the order of the blow-ups is important, as explained in the following. Assume that C_1 is blown-up first. Let D_1, D_2 denote the exceptional divisors, which are P^1 rulings over C_1, C_2 and let E_1, E_2 denote the classes of the fibers. Also, let D_1', D_2' denote the strict transforms of the vertical divisors p^*C_1, p^*C_2. Near the intersection point $P \in S_0$, the geometry looks as in fig. 2.

Note that when performing the second blow-up along C_2, the exceptional divisor D_1 undergoes an embedded blow-up at the point P. This results in a reducible fiber with three components, the third component E_3 being the difference $E_1 - E_2$. The strict transform D_2' also acquires a reducible fiber with a component identified with E_3. Therefore we have $D_1 \cdot D_2' = E_3$. Note that E_3 is then a (-1) curve in both surfaces, hence we can compute its normal bundle

$$N_{E_3/\bar{B}_3} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)\,. \quad (4.1)$$

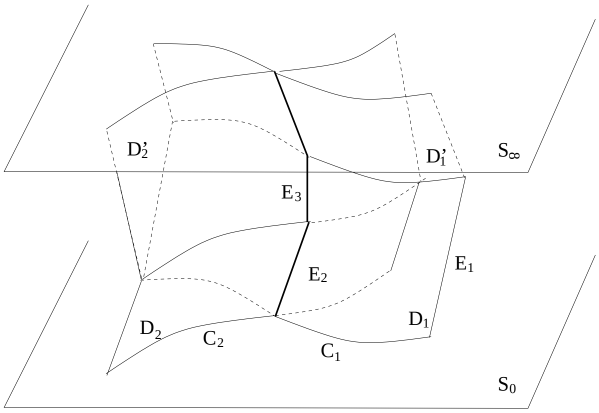

Figure 2: The blow-up geometry corresponding to two intersecting horizontal fivebranes. Note that the fiber acquires a third component $E_3 = D_1 \cap D_2'$ over the intersection point.

If the blow-ups are performed in different order, the roles of D_1, D_2 are interchanged, so that that the extra component E_3 lies now in D_2, D_1'. In this case, $E_3 = E_2 - E_1$. In fact the two models are related by a flop transition whose physical interpretation will be investigated below.

At classical level, the map between heterotic and F-theory parameters is similar to the six dimensional case. The sizes of E_1, E_2 give the positions ϕ_1, ϕ_2 of the fivebranes in the M-theory interval, and are therefore generically different. Furthermore, the size of E_3 is related to the distance $|\phi_1 - \phi_2|$ between the two fivebranes along the M-theory interval. In particular, note that as long as E_3 is of finite size, the two fivebranes do not really intersect each other. Finally, as noted above, there are two choices for the curve E_3, namely $E_1 - E_2$ or $E_2 - E_1$. The process of going from one situation to the other can be interpreted as moving the two fivebranes past each other along the M-theory interval, with the fivebranes intersecting when $E_1 = E_2$, i.e. when E_3 shrinks to zero size.

Thus, we find that intersecting fivebranes on the heterotic side correspond to an isolated singularity in the base of the F-theory fourfold. If both curves are rational as stated above, the singularity in B_3 can be very simply identified as a conifold by looking at the toric diagram of the fan of the fourfold base.

The two resolutions of the singularity are then different triangulations of the fourfold, related by a flop, which is precisely the transition of E_3 from $E_1 - E_2$ to $E_2 - E_1$. It is worth emphasizing here that the conifold singularity is not in the

Calabi–Yau fourfold itself, but in the base $\bar{B}_3$. Alternatively, the normal bundle computation (4.1) shows that the singularity resulting from shrinking E_3 is a simple node. Thus, separating the fivebranes on the M-theory interval, which gives E_3 a non-zero volume, corresponds to resolving the conifold singularity, and the two ways of moving them apart correspond to the two possible small resolutions of the conifold.

The relation between intersecting fivebranes and the conifold singularity has been well studied recently [25, 26, 27, 28, 29, 30]. It has been shown there that starting from a configuration of two infinite Type IIA NS-fivebranes stretching along (012345) and (012389), and T-dualizing along x_6, we obtain Type IIB on a conifold. The separation of the NS-fivebranes along x_6 maps to a component of the B field under the T-duality. Furthermore, if we have fourbranes along (01236), these map to threebranes on the conifold singularity. In our case, we have fivebranes in the heterotic theory. Thus we can heuristically relate our situation with the IIA case by first going to M-theory by taking the strong coupling limit of the IIA configuration (in which case the fourbranes now become fivebranes along (01236&10)), and then compactifying x_7 to an interval ($S^1/\mathbb{Z}_2$). We emphasize that this is only heuristic, since the analysis of refs. [25, 27, 28, 29, 30] is valid only for infinite branes, while the fivebranes here are compact (at least along the wrapped curves). However, since the geometric description of the heterotic/F-theory vacua above is strictly valid only when both the volumes of the section σ (and hence the wrapped curves) and the elliptic fiber are both large, we have reason to trust our conclusions. In particular, we see that in the F-theory case, we cannot turn on B fields (as they are not $\mathrm{SL}(2,\mathbb{Z})$ invariant), so this is analogous to having $x_6 = 0$ in the IIA case. Moreover, we see that there is a very suggestive relation between the vertical heterotic fivebranes and the IIA fourbranes, since in the M-theory limit, the fourbranes become fivebranes wrapping (6&10), which form a torus (since both x_6 and x_{10} are compactified on circles), and the vertical heterotic fivebranes wrap the elliptic fibre, which is also a torus. Furthermore, in both cases, they are related to threebranes on the dual.

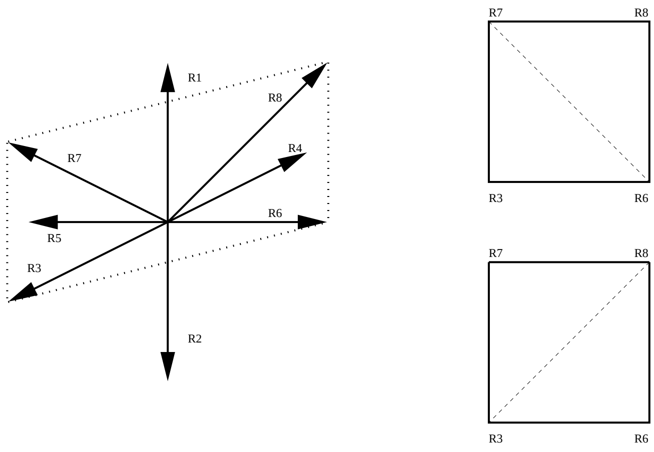

Figure 3: Toric diagram of a conifold. The figure on the left gives the fan of $(P^1)^3$ blown up along two P^1's in the base $(P^1)^2$. The fan of the base $(P^1)^2$ is described by the rays $R3, \ldots, R6$. The conifold singularity is described by the rays $R3, R6, R7$ and $R8$. The two small resolutions of the conifold correspond to different triangulations of the fan, and are shown on the right.

We can use the analogy developed above to describe the situation when the F-theory threebranes coincide with the conifold singularity in the base. Note that the threebranes only move along $\bar{B}_3$, or rather, a section of the elliptic Calabi–Yau fourfold X_4. This follows from viewing the F-theory vacuum as Type IIB compactified on $\bar{B}_3$, since the threebranes and sevenbranes are, strictly speaking, Type IIB objects. When n threebranes sit on the node of the conifold (recall that it is really the base $\bar{B}_3$ which develops the conifold singularity), the heterotic dual consists of intersecting horizontal fivebranes with n vertical fivebranes at the point of intersection. Thus, we are again in a situation similar to the one discussed in [25, 27, 28, 29, 30], where the M-theory dual of n threebranes on a conifold consists of n fivebranes along (01236&10) between two transverse fivebranes along (012345) and (012389), respectively. However, since we are really in F-theory, we must switch off the the B and $\tilde{B}$ fields. This corresponds in the M-theory picture to setting the x_6 and x_{10} separations of the transverse fivebranes to zero, so that there really is only one set of fivebranes between them, and not two. Thus while the gauge theory in the more general situation of [25, 27, 28, 29, 30] is an $\mathrm{SU}(n) \times \mathrm{SU}(n)$ theory, we see that in our case, we only get an $\mathrm{SU}(n)$ gauge theory at the singularity. Note that unlike the situation in section 3.2, the arguments of [23] showing the existence of an interacting theory on the fivebrane intersection are supported by the F-theory picture.

The relation between the present situation and the one discussed in [25, 27, 28, 29, 30] suggests a useful interpretation of the resolution of the conifold singularity in terms of the fivebranes. We note that the size of E_3, the resolving P^1, is related to the separation the fivebranes along the M-theory interval. We have argued before that this corresponds to the x_7 direction in the picture of [25, 27, 28, 29, 30]. Therefore, resolving the conifold should correspond to separating the transverse fivebranes along x_7 in the brane picture. In fact, separating the fivebranes along x_7 corresponds to turning on a Fayet-Iliopoulos D-term in the lagrangean of the $N = 1$ U(1) gauge theory describing the conifold, whose coefficient ζ is the x_7 separation of the branes [25, 31]. But, as shown in [25], setting $\zeta \neq 0$ gives a resolution of the conifold singularity, and in fact there are two different resolutions depending on the sign of ζ. Moving the fivebranes past each other in the x_7 direction thus corresponds to a flop transition in the dual theory, exactly as in the heterotic picture above.

The conifold has another interesting property: it can be deformed to a smooth manifold. Thus, we expect to be able to deform the base $\bar{B}_3$ away from the conifold. What is the heterotic dual of this deformation?

It was shown in [32] that the conifold metric is deformed to a smooth metric when threebranes sit on the node of the conifold. We will show in our case that even in the absence of the threebranes, there are often deformations that smooth out the singularity.

Consider the following example, first discussed in [4]. We consider the heterotic theory on the Calabi–Yau threefold elliptically fibered over the Hirzebruch surface

F_1, with E_8 bundles chosen so that the dual Calabi–Yau fourfold is elliptically fibered over $F_{100} = F_1 \times P^1$. Now let us consider two horizontal fivebranes, one wrapping the zero section C_0 and the other wrapping the fiber class f of F_1. Since $C_0 \cdot f = 1$, these fivebranes intersect exactly once, therefore we are in the situation discussed above. Setting the size of the exceptional component E_3 to zero corresponds to moving the fivebranes on top of each other in the M-theory interval so that they actually intersect, and we obtain a conifold singularity in $\bar{B}_3$. However, we can now deform away from the conifold as follows. The infinity section C_∞ of F_1 is in the same divisor class as $C_0 + f$, but has one additional deformation modulus (C_0 has no moduli as it has negative self-intersection and so cannot be moved, f has exactly one modulus, and C_∞ has two moduli). Thus, we can deform the intersecting fivebranes into a single one wrapping C_∞. On the F-theory dual, we obtain a Calabi–Yau fourfold with different Hodge numbers (since the moduli are different) but the same Euler number (since the number of threebranes is the same).

On the other hand, if the heterotic base were the Hirzebruch surface F_2 instead, then we cannot deform the two curves C_0 and f into a single curve. Thus the node at the point of intersection of C_0 and f cannot be deformed. We see therefore that there is a global obstruction to the deformation of the conifold singularity obtained by the intersection of the two curves. This global obstruction can be simply stated as follows. The number of moduli of a curve C in the heterotic base B_2 is given by $h^0(N_{C/B_2}) = h^0(\mathcal{O}_C(C))$, the number of deformations of the normal bundle $N_{C/B_2} = \mathcal{O}_C(C)$ of C in B_2. When $h^{1,0}(B_2) = h^1(B_2, \mathcal{O}) = 0$, the number of moduli is also the dimension of the linear system of the divisor associated with the curve C, which is simply $h^0(B_2, \mathcal{O}(C)) - 1$. Clearly, if we have two curves C_1 and C_2, then we can deform to a single curve only if $C_1 + C_2$ has more moduli than C_1 and C_2, so that the deformation exists only when

$$h^0\left(\mathcal{O}_{(C_1+C_2)}(C_1 + C_2)\right) > h^0(\mathcal{O}_{C_1}(C_1)) + h^0(\mathcal{O}_{C_2}(C_2))\,. \tag{4.2}$$

To summarize, we have found that intersecting horizontal fivebranes are dual to isolated conifold singularities in the F-theory base. At classical level, the theory exhibits two branches corresponding to the small resolution and respectively to the deformation of singularities. The passage from one branch to the other represents an extremal transition in the fourfold geometry. Although the situation seems similar to the conifold singularities encountered in $N = 2$ string vacua [33, 34], there are important differences. First note that the S^3 cycle present in the local deformation of the conifold is in general homologically trivial in the deformed F-theory base. If C_1, C_2 are rational curves, this follows easily by noting that the third Betti number of the deformed base is zero. Therefore, the usual restrictions on Kähler resolutions imposed by the presence of three-cycles are absent in this case. In particular, small Kähler resolutions of single isolated conifold are allowed.

Physically, this means that on the deformation branch, we cannot identify a state of vanishing mass at the singularity. Moreover, even if S^3 were homologically nontrivial, a wrapped threebrane would still not define a stable BPS state due to the reduced amount of supersymmetry. At the same time, a threebrane wrapped around the S^2 cycle of a small Kähler resolution will give rise to a stable BPS string whose tension goes to zero as we approach the singularity. In the context of Calabi-Yau conifolds, this string has been identified with a flux tube between charged sources in the confining phase of an $N = 2$ gauge theory [35]. Here, single conifold transitions are allowed, therefore such an interpretation is no longer valid. Instead, we can regard the stable tensionless string as (weak) evidence for the existence of an interacting superconformal theory localized at the singularity. This would be dual to the theory on intersecting heterotic fivebranes.

Thus far, we have only considered curves of genus zero, i.e. P^1's. Let us now see how this discussion needs to be modified when we have curves of higher genus. First, according to section 2, the F-theory dual of a fivebrane wrapping a curve of genus g is once again a blowup of B_3 over the same curve, the size of the blowup giving the position of the fivebrane in the M-theory interval. When two such curves intersect in n points, the local geometry near each intersection point is identical to that represented in fig. 2. Therefore we obtain n exceptional P^1 components of type E_3 which are in the same cohomology class, either $E_1 - E_2$ or $E_2 - E_1$. Since the total space is Kähler, they are also of the same size giving the separation between the fivebranes on the interval. When the two fivebranes coincide, all the exceptional P^1 components shrink simultaneously and we obtain n isolated conifold singularities. This picture shows that the n isolated singularities cannot be resolved independently. There are precisely two ways of resolving the singularity (which are related by a flop), corresponding to the two ways of separating the fivebranes on the interval. Once again, deforming the two curves C_1 and C_2 into a single curve in the class $C_1 + C_2$ is a deformation away from the singularity. The global obstruction to this is again given by eq. (4.2).

4.2 Parallel fivebranes

Now consider the situation when we have two horizontal fivebranes wrapping the same curve C. Again, we first consider the case when the curve in question is a P^1. On the F-theory side, the base $\bar{B}_3$ now has two intersecting exceptional P^1's E_1, E_2 fibered over C (E_3 is the proper transform of the original fiber) with the size of the E_2 giving the mutual separation of the fivebranes along the M-theory interval. Blowing down this P^1, therefore, corresponds to moving the fivebranes on top of each other. This gives an A_1 singularity over C in $\bar{B}_3$. More generally, if we have k coincident fivebranes wrapping C, we would get an A_{k-1} singularity over C in $\bar{B}_3$. In addition, if we have n coincident vertical fivebranes, this corresponds on the F-theory side to n threebranes on the A_{k-1} singularity. As before, we see that this is very similar to the situation discussed in refs. [29, 28], except that the B fields are zero in our situation, corresponding to zero separation of the branes along x_6 and x_{10} in the brane description. For clarity, we will restrict our attention to the case when we have only two parallel fivebranes for the rest of this section. The generalization to more branes is elementary.

Apart from the obvious resolution of this singularity (blowing up the P^1 again, corresponding to separating the fivebranes), we can deform away from it as follows. First, if C is movable (i.e. if $h^0(\mathcal{O}_C(C)) > 0$), then we can deform away from the singularity by moving one curve away from the other in B_2. If $C.C = 0$, then the two curves no longer intersect, and the corresponding $\bar{B}_3$ is smooth. However, if $C \cdot C > 0$ ($C \cdot C < 0$ is impossible here, since we have assumed that C is movable), then the curves intersect in a set of points, and $\bar{B}_3$ becomes a conifold, as in the previous section, with a set of isolated nodes. We can then either resolve the conifold by blowing it up, i.e. separating the fivebranes, or by deforming it further. This deformation is only possible if $2C$ has a smooth section different from C, i.e. if $h^0(\mathcal{O}_{2C}(2C)) > 2h^0(\mathcal{O}_C(C))$. In this case, the curves can be deformed into a single curve, and we get a single fivebrane wrapping a smooth curve in the class $2C$. Of course, in this case, we could have directly deformed the two curves into a single curve without passing through the conifold phase. If C is not movable, then we cannot deform to a conifold. However, if $h^0(\mathcal{O}_{2C}(2C)) > 2h^0(\mathcal{O}_C(C))$, then we can still deform to a single curve, because $2C$ is then movable. If, however, $2C$ is also not movable, then we cannot deform away at all. The geometrical analysis can be easily generalized to curves of higher genus.

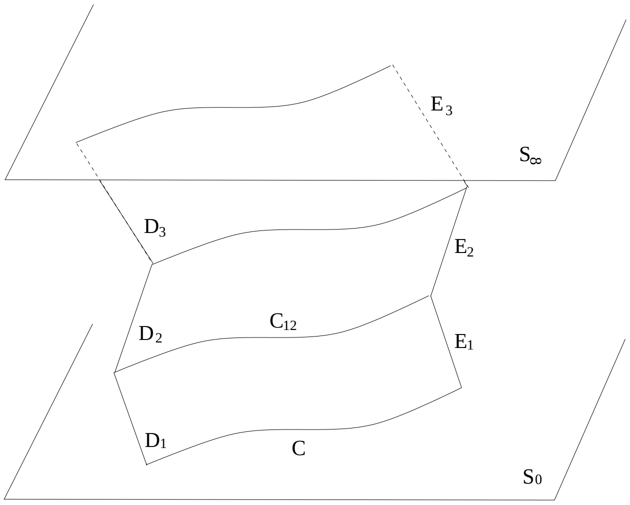

Figure 4: The blow-up geometry corresponding to two fivebranes wrapping the curve C.

Note that in contrast with the previous cases, the low energy theory obtained here is essentially associated to a curve of A_{k-1} singularity in Type IIB theory. Therefore we will have an interacting theory which can be described as the compactification of the $(2,0)$ field theory on a Riemann surface. If vertical fivebranes are added to the picture, the dual F-theory phenomenon consists of threebranes transverse to a curve of A_{k-1} singularities in IIB theory. This is a familiar situation encountered many times in the literature, starting with [29]. However, the resulting effective theory may present some complications due to the finite size of the singular curve. This will be discussed in the next subsection.

Finally, we can combine the situation here with that of the previous subsection, and consider k_1 horizontal fivebranes wrapping C_1, k_2 horizontal fivebranes wrapping C_2, and n vertical fivebranes at a point of intersection of C_1 and C_2. First, ignoring the vertical fivebranes, the dual F-theory degeneration consists of two curves of A_{k_1-1} and A_{k_2-1} singularities intersecting transversely in $\bar{B}_3$. This results in a nonabelian conifold singularity [36, 37]. Near each intersection point, this is again similar to a situation considered in ref. [28], where the singularity was shown to be of the form $xy = z^{k_1}w^{k_2}$. Note again that the x_6 separation of the branes is zero since we have to set the B fields to zero. It is an easy exercise to describe geometrically the various deformations and resolutions of this situation. The specific example of $k_1 = 1, k_2 = 2$ was worked out in 4 of ref. [28]. Bringing the k vertical fivebranes near an intersection point corresponds to placing n threebranes at a nonabelian conifold singularity in F-theory. The description of the low energy effective theory is quite difficult in this case, as will be detailed in the following.

4.3 Merging horizontal and vertical fivebranes — a second puzzle

Consider the situation discussed in the previous subsection, namely n vertical fivebranes intersecting $k > 1$ horizontal fivebranes wrapped around a curve C in B_2. Here we encounter a qualitatively new phenomenon. The fivebranes can merge together into a single fivebrane wrapping a smooth curve C' in the Calabi-Yau space Z. An obvious necessary condition for this is $(kC + nf) \cdot C > 0$, i.e. $n > -kC \cdot C$. The resulting fivebrane is, in a sense, "skew". The case $k = 1$ was discussed in section 3.2.

Therefore, it follows that there is a new branch of the theory along which the low energy effective action consists of $g(C')$ abelian gauge fields whose couplings are governed by the jacobian of C'. The main problem is to understand the F-theory origin of this new branch. As a first step, note that this phenomenon cannot admit a pure geometrical interpretation. This is because horizontal and vertical fivebranes map to very different objects in F-theory. The horizontal fivebranes are described in terms of blowup modes in the base, which can be described purely in terms of the fourfold geometry, but the vertical fivebranes map to threebranes, which are not related in any way to the geometry - rather, they constitute additional non-geometric data necessary for the complete description of the F-theory vacuum. Therefore, when these two kinds of fivebranes merge into a single fivebrane, it is unlikely to find a pure geometric description.

In the following, we suggest a possible resolution of this puzzle based on a careful analysis of threebranes transverse to a curve of A_{k-1} singularities. In principle, threebranes transverse to an A_{k-1} singularity are described by an A_{k-1} quiver gauge theory [29]. The space time parameters — blow-up modes and theta angles — are realized as FI terms in the brane gauge theory. However, at the same, the spacetime moduli are coordinates along the flat directions of the A_{k-1} (2,0) theory localized at the singularity. A similar situation is encountered in the $(1,0)$ theories with tensor multiplets discussed in [39, 40]. In the present situation, the spacetime $(2,0)$ is compactified on a Riemann surface of finite size, therefore it yields an interacting four dimensional theory with no lagrangean description. Moreover, the $(2,0)$ degrees of freedom must interact nontrivially with the threebrane degrees of freedom, as a result of the previous interplay between space time moduli and brane gauge theory. Therefore, we conclude that the theory associated to F-theory threebranes transverse to a curve of A_{k-1} singularities must be a complicated interacting fixed point. The new fivebrane branch found above can then be interpreted as a low energy Coulomb branch emerging from this fixed point. In particular, it is a non-geometric branch.

When $C \cdot C = -2$, $\Sigma = \pi^* C \simeq P^1 \times T^2$, and there is a brane construction for the above situation. The above Coulomb branch can then be interpreted in terms of fractional branes [41]. It is not clear if this description holds in the more general cases.

Another possibility [42], is to compactify on a further circle to M-theory. In this picture, the F-theory threebranes map to membranes. Deforming $kC + nf$ to a single irreducible curve might correspond to absorbing n membranes and turning on a four-form field strength flux carrying n units of membrane charge. Lifting back to F-theory, this would map to turning on $H, \tilde{H}$ fluxes. We are currently investigating this possibility and hope to report on it in the future.

Finally, note that the situation is even more complicated when vertical fivebranes sit at a point of intersection of multiple horizontal fivebranes.

5. Discussion of the superpotential

In the previous sections, we have studied the $N = 1$ moduli space from an essentially classical point of view. Here we will discuss how the classical picture is modified by nonperturbative instanton effects. From the heterotic perspective, there can be various nonperturbative phenomena generated by spacetime or worldsheet instantons. All these effects have a simple and unitary F-theory interpretation in terms of threebrane instantons wrapping divisors in the threefold base. In the following we will give a systematic treatment of these effects for the small instanton degenerations considered so far. Note that in order obtain correct results, the zero mode computations must be performed away from the zero degeneration limit, that is for a smooth elliptic Calabi-Yau fourfold $\pi : X \to B_3$. This can be seen, for example, by taking into account the M-theory origin of the instantons explained in [8]. In order to obtain such a smooth model, the base $\bar{B}_3$ must be blown-up a number of times along certain curves contained in the two II^* sections. As before, let $\bar{B}_3$ denote the blown-up base. Many of the results of this section have been obtained independently by A. Grassi [43]. We are grateful to her for sharing her results with us.

i) *Heterotic worldsheet instantons.* Generically, these correspond to threebranes wrapping divisors $W \subset \bar{B}_3$ which are vertical with respect to the map $p : \bar{B}_3 \to B_2$ [8]. Here, this is still valid if W is the pull back of a generic curve C in the base, other than the support of the exceptional locus. In this case, we can apply directly the results of [9]. The number of zero modes is given by

$$\chi\left(\mathcal{O}_{\pi^* W}\right) = \frac{1}{2} K_{\bar{B}_3} W^2 \, . \tag{5.1}$$

Since $W = p^* C$, $W^3 = 0$ and the adjunction formula

$$K_W = (K_{\bar{B}_3} + W) \cdot W \, , \tag{5.2}$$

shows that

$$K_{\bar{B}_3} W^2 = K_W \cdot W = -2(C^2)_{B_2} \, . \tag{5.3}$$

Therefore, only divisors supported on curves with $(C^2)_{B_2} = -1$ can contribute. As the base $B_2 \simeq F_n$, $0 \leq n \leq 2$, it follows that a superpotential can be generated only if $n = 1$. The analysis of [8, 9] shows that in this case, a superpotential is actually generated.

ii) *Noncritical string instantons.* Regarding the heterotic theory as M-theory on an interval, there are two types of noncritical strings corresponding to membranes stretching between two fivebranes or membranes stretching between a fivebrane and a nine dimensional wall. Some effects associated with these instantons have been discussed in [44, 45]. These are BPS strings whose tension is proportional to the separation between the branes. Therefore, taking into account the moduli map developed in the previous sections, they can be naturally identified with Euclidean threebranes wrapping the exceptional divisors in F-theory. More precisely, let us consider a configuration of k fivebranes wrapping a curve C in B_2 and separated along the interval. The corresponding F-theory geometry consists of a sequence of divisors $D_1, D_2 \ldots D_k, D_{k+1}$ with normal crossings as explained in section 4.2. Threebranes wrapped on the divisors D_1, D_{k+1} correspond to membranes stretching between the first and the last fivebrane and the nine dimensional walls respectively. At the same time, threebranes wrapped on $D_1 \ldots D_k$ correspond to membranes stretching between consecutive fivebranes. The number of zero modes can be computed again using formula (5.1). In fact a simple local computation shows that the fibers E_1, E_{k+1} of D_1, D_{k+1} are negative extremal rays in the blown-up threefold base, therefore we can apply the results of [9] (Ex. 2.9.1 and Prop. 3.4). This shows that

$$\chi\left(\mathcal{O}_{\pi^* D_1}\right) = \chi\left(\mathcal{O}_{\pi^* D_{k+1}}\right) = 1 - g(C) \, , \tag{5.4}$$

where $g(C)$ is the genus of the curve. Moreover, it can be shown as in [9] that the Hodge numbers of these divisors are

$$h^{0,0} = 1, \qquad h^{0,1} = g(C)\,, \qquad h^{0,2} = h^{0,3} = 0\,. \tag{5.5}$$

Therefore the divisors D_1, D_{k+1} contribute if and only if C is a rational curve. This formula does not apply to the middle divisors $D_1 \ldots D_k$ since their fibers are not negative extremal rays. The number of zero modes can be computed recursively as follows. For simplicity consider the case $k = 2$ represented in fig. 4. Let n_i, $i = 1, 2, 3$ denote the degrees of the rulings of the three divisors. Note that by construction,

$$n_1 = n_2 + (C^2)_{B_2} \tag{5.6}$$

and the divisors D_1, D_2 intersect along a common section C_{12} isomorphic to C. The adjunction formula (5.2) shows that

$$K_{\bar{B}_3} D_2^2 = \left(K_{D_2} - D_2^2\right) \cdot D_2\,, \tag{5.7}$$

where

$$K_{D_2} = -2C_{12} + (2g(C) - 2 - n_2)E_2\,. \tag{5.8}$$

Furthermore, a local computation shows that the restriction of the normal bundle $N_{D_2/\bar{B}_3}$ to the P^1 fiber E_2 is of degree -2. Therefore, we have

$$N_{D_2/\bar{B}_3} = -2C_{12} + aE_2\,, \tag{5.9}$$

where a is an integer number which can be determined as follows. The triple intersection $D_1 D_2^2$ can be computed in two equivalent ways

$$\begin{aligned} D_1 D_2^2 &= (C_{12}^2)_{D_1} = n_1 \\ &= C_{12} \cdot D_2 = 2n_2 + a\,. \end{aligned} \tag{5.10}$$

Using also (5.6), we find

$$a = (C^2)_{B_2} - n_2\,. \tag{5.11}$$

Finally, (5.7), (5.8) and (5.9) imply that

$$\chi\left(\mathcal{O}_{\pi^* D_2}\right) = -(C^2)_{B_2} + 2g(C) - 2 = -C \cdot K_{B_2}\,. \tag{5.12}$$

It is clear that this recursive step can be applied for each of the divisors $D_2 \ldots D_k$ yielding the same result. The Hodge numbers can be computed using the general expressions in [9] and the Riemann-Roch theorem on the D_i

$$h^{0,0} = 1\,, \qquad h^{0,1} = g(C)\,, \qquad h^{0,2} = -C \cdot K_{B_2} + g(C) - 1\,, \qquad h^{0,3} = 0\,. \tag{5.13}$$

JHEP06(1999)002

Note that again there is an exceptional case, namely $B \simeq F_2$, $C = C_0$ when the above general formulae do not hold. In that case, we obtain similarly

$$h^{0,0} = 1, \qquad h^{0,1} = 1\,, \qquad h^{0,2} = h^{0,3} = 0\,. \tag{5.14}$$

Therefore, although $-C \cdot K_{B_2} = 1$ is a necessary condition for the generation of a superpotential, it may not be sufficient [8]. A sufficient condition also requires $g(C) = 0$, showing that the contribution is certain for rational curves satisfying $(C^2)_{B_2} = -1$. If the base is a Hirzebruch surface of degree $0 \leq n \leq 2$, this can be realized only when $n = 1$ and C is a rational curve.

iii) *Exceptional Instantons.* In addition to the effects discussed so far, there are also contributions coming from the resolution of the sections of E_8 singularities of the elliptic fibration. These have been shown in [15, 16] to give the expected gauge theoretic nonperturbative superpotentials and will not be discussed here.

5.1 Toric description

It is interesting also to consider the computation of the zero modes in the light of toric geometry, which provides a method of explicitly constructing Calabi–Yau fourfolds as hypersurfaces in toric varieties. Such a Calabi–Yau fourfold X is described by a dual pair of five dimensional reflexive polyhedra (Δ, ∇). In particular, the points in ∇ correspond to divisors in the Calabi–Yau manifold. (See, for example, [46] for a useful review). For any point p in ∇, the arithmetic genus of the corresponding divisor in the Calabi–Yau fourfold is given by a formula of [47]. If p is interior to a face Θ^*_p in ∇, and Θ_p is the dual face in Δ (defined by $\Theta_p = \{q \in \Delta| < q, v >= -1, \forall v \in \Theta^*_p\}$, note that $\dim(\Theta_p) + \dim(\Theta^*_p) = 4$), and $l(\Theta)$ is the number of lattice points interior to Θ, the formula of [47] is

$$\chi(D_p, \mathcal{O}(D_p)) = 1 - (-1)^{\dim(\Theta_p)} l(\Theta_p)\,, \tag{5.15}$$

where D_p is the divisor in X corresponding to the point p. (If $\dim(\Theta^*_p) = 4$, the point p corresponds to a divisor in the embedding variety that does not intersect the space X, and hence does not give a divisor on X).

Since we are studying F-theory compactifications, X is an elliptic fibration, i.e. there is a projection $\pi : X \to B_3$ whose generic fiber is an elliptic curve $\mathcal{E}$, and the vertical divisors (which project down to divisors in B_3) are the only ones that contribute to the superpotential. Now, by theorems in [48, 45], the polyhedron ∇ contains a slice $\nabla_{\mathcal{E}}$ through the origin, where $\nabla_{\mathcal{E}}$ is the polyhedron describing the elliptic fiber (which is a torus, and hence Calabi–Yau) as a hypersurface in a toric variety. The theorem also assures us that there exists a projection acting on ∇, which projects $\nabla_{\mathcal{E}}$ to a point, and whose image is the fan Σ_{B_3} of B_3. Thus, points in ∇ project down to points in Σ_{B_3}, and hence describe divisors in B_3. Furthermore, in order that a heterotic dual exist, X must also admit a $K3$ fibration that is consistent with the elliptic fibration structure. This implies that B_3 is itself a P^1 fibered over B_2, the base of the heterotic threefold Z.

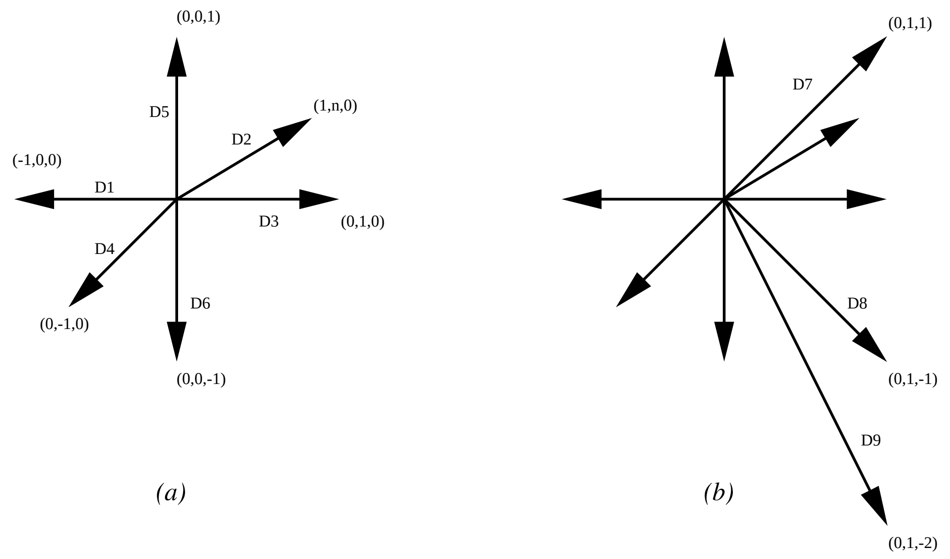

Figure 5: In (a), we have the fan of $F_{n,0,0} = F_n \times P^1$, the base of the fourfold dual to the heterotic compactification on the elliptic Calabi–Yau threefold with base F_n. In (b), we have the fan of $F_{n,0,0}$ blown up three times over the zero section C_0 of F_n, corresponding to three fivebranes wrapping C_0 in the heterotic base F_n.

When k fivebranes wrap a curve C in B_2 which corresponds to a toric divisor, the corresponding base $\bar{B}_3$ acquires k blowup modes over the corresponding divisor (see fig. 5). Thus, we get a line of points $p_1, p_2, \ldots, p_{k+1}$ all of which project down to the divisor C in B_2 under the projection $P : \bar{B}_3 \to B_2$. The fact that all of these points lie in a straight line indicates that they all lie in the same face of ∇. In the simplest cases, p_1 and p_{k+1} are vertices in ∇, while $p_2, \ldots, p_k$ lie in the edge joining p_1 and p_{k+1}. In general, $p_2, \ldots, p_k$ are interior to a face Θ^* of ∇, while p_1 and p_{k+1} will usually lie on the boundary of Θ^*, and hence in faces of lower dimension. Now the expression (5.15) yields the same result for all points interior to a given face Θ^* of ∇. Thus we see immediately that $p_2, \ldots, p_k$ all have exactly the same arithmetic genus, and therefore, either they all contribute to the superpotential or they all do not. Moreover, it follows that the points p_1 and p_{k+1} can have a different arithmetic genus.

We can actually compute the arithmetic genus of the divisors in the fourfold that correspond to heterotic worldsheet instantons and noncritical string instantons as follows. The points of ∇ for the fourfold X can be put in the form (x, y, z, u, v), such that the slice $(0, 0, 0, u, v)$ gives $\nabla_{\mathcal{E}}$, and the slice $(0, 0, z, u, v)$ gives ∇_{K3}. The projection $P_3 : (x, y, z, u, v) \to (x, y, z)$ yields the fan Σ_{B_3} of B_3, the base of the elliptic fibration, while $P_2 : (x, y, z, u, v) \to (x, y)$ gives the fan Σ_{B_2} of the base B_2 of the $K3$ fibration. The points $q \in \Delta$ satisfy $< q, p > \geq -1, \forall p \in \nabla$. In terms of Δ, the polyhedron Δ_{K3} of the $K3$ fiber is seen as a projection $\Pi_{K3} : \{a, b, c, d, e\} \to \{c, d, e\}$, while $\Delta_{\mathcal{E}}$ of the elliptic fiber is given by $\Pi_{\mathcal{E}} : \{a, b, c, d, e\} \to \{d, e\}$[48, 45]. (We will use round brackets to denote points in ∇, and curly brackets to denote points in Δ.) For example, the polyhedron given by the points

$$(1,1,0,2,3),(0,1,0,2,3),(0,0,1,2,3),(0,0,0,2,3),(0,0,0,1,2),(0,0,0,1,1),$$
$$(0,0,0,0,1),(0,0,0,0,0),(0,0,0,0,-1),(0,0,0,-1,0),(-1,0,0,2,3),$$
$$(0,-1,0,2,3),(0,0,-1,2,3)$$

describes a fourfold elliptically fibered over $F_1 \times P^1$, which can also be viewed as a $K3$ fibration over F_1.

Consider first the case of the worldsheet instanton. This corresponds to a point $p \in \nabla$ that projects down to a point $\tilde{p} \in \Sigma_{B_2}$ under P_2, which describes a divisor (curve) $C_{\tilde{p}}$ in B_2. Now $C_{\tilde{p}} \cdot C_{\tilde{p}} \geq -2$, otherwise the elliptic fibration will be singular over $C_{\tilde{p}}$. In such a case we do not have heterotic worldsheet instantons, but rather instantons associated with the exceptional fibers. For $C_{\tilde{p}} \cdot C_{\tilde{p}} > -2$, the point p is a vertex, whereas for $C_{\tilde{p}} \cdot C_{\tilde{p}} = -2$, p is interior to an edge. Now, the points q which are *interior* to a face Θ in Δ satisfy $\langle q, v \rangle = -1, \forall v \in \Theta^*$ and $\langle q, v \rangle \geq 0, \forall v \notin \Theta^*$ $(v \in \nabla)$. But ∇ contains the reflexive polyhedron ∇_{K3} as a slice. The only point in Δ_{K3} that has non-negative product with *all* the points in ∇_{K3} is the origin $\{0, 0, 0\}$, since, by reflexivity, it is the unique interior point of Δ_{K3}. Since $\Pi_{K3}(\Delta) = \Delta_{K3}$, it follows that the only points in Δ that have non-negative product with all the points in $\nabla_{K3} \subset \nabla$ are of the form $\{a, b, 0, 0, 0\}$. Now, all the points interior to Θ_p must have non-negative product with any point in ∇ not in Θ_p^*. In particular, they have non-negative product with any point in $\nabla_{K3} \subset \nabla$, and so must be of the form $\{a, b, 0, 0, 0\}$. It is then a simple matter to count $l(\Theta_p)$, the number of points interior to Θ_p.

For $C_{\tilde{p}} \cdot C_{\tilde{p}} > -2$, we find $l(\Theta) = 1 + C_{\tilde{p}} \cdot C_{\tilde{p}}$. Since p is a vertex, $\dim(\Theta_p) = 4$, and (5.15) gives

$$\chi(D_p, \mathcal{O}(D_p)) = -C_{\tilde{p}} \cdot C_{\tilde{p}}\,.$$

Using the results of [47] we also find

$$h^{0,0} = 1\,, \qquad h^{1,0} = 0\,, \qquad h^{2,0} = 0\,, \qquad h^{3,0} = 1 + C_{\tilde{p}} \cdot C_{\tilde{p}}\,,$$

so that a superpotential is generated if and only if $C_{\tilde{p}} \cdot C_{\tilde{p}} = -1$.

When $C_{\tilde{p}} \cdot C_{\tilde{p}} = -2$, we get $l(\Theta) = 1$. But $\dim(\Theta_p) = 3$ (since p must lie in an edge, otherwise the fourfold X will not admit a $K3$ fibration consistent with the elliptic fibration structure), so (5.15) gives

$$\chi(D_p, \mathcal{O}(D_p)) = 2 = -C_{\tilde{p}} \cdot C_{\tilde{p}}\,.$$

In fact, we find

$$h^{0,0} = 1, \qquad h^{1,0} = 0\,, \qquad h^{2,0} = 1, \qquad h^{3,0} = 0\,.$$

Now consider the case of the noncritical string instantons. We now have a line of points $p_1, \ldots, p_{k+1} \in \nabla$ all of which project down to the same point $\tilde{p} \in \Sigma_{B_2}$. Once again, $C_{\tilde{p}} \cdot C_{\tilde{p}} \geq -2$, otherwise we have exceptional fibers.

For $C_{\tilde{p}} \cdot C_{\tilde{p}} > -2$ the points p_j, $2 \leq j \leq k$ lie in the edge $\Theta^*_{1,k+1}$ joining p_1 and p_{k+1}. Once again, we see that all the points interior to $\Theta_{1,k+1}$ are of the form $\{a, b, 0, 0, 0\}$, and $l(\Theta_{1,k+1}) = 1 + C_{\tilde{p}} \cdot C_{\tilde{p}}$. Thus, for the points p_j, $2 \leq j \leq k$, we get, using $\dim(\Theta_{1,k+1}) = 3$ and (5.15),

$$\chi(D_{p_j}, \mathcal{O}(D_{p_j})) = 2 + C_{\tilde{p}} \cdot C_{\tilde{p}} = -K_{B_2} \cdot C_{\tilde{p}}\,.$$

We also find

$$h^{0,0} = 1\,, \qquad h^{1,0} = 0\,, \qquad h^{2,0} = 1 + C_{\tilde{p}} \cdot C_{\tilde{p}} = -K_{B_2} \cdot C_{\tilde{p}} - 1\,, \qquad h^{3,0} = 0\,.$$

But $\Theta_{1,k+1}$ is the common face of Θ_1 and Θ_{k+1}. We see that all the points of the form $\{a, b, 0, 0, 0\}$ in Θ_1 lie in $\Theta_{1,k+1}$ and hence cannot be interior to Θ_1. Therefore, $l(\Theta_1) = 0$, and similarly, $l(\Theta_{k+1}) = 0$. We thus get

$$\chi(D_{p_1}, \mathcal{O}(D_{p_1})) = \chi(D_{p_{k+1}}, \mathcal{O}(D_{p_{k+1}})) = 1\,,$$

and moreover,

$$h^{0,0} = 1\,, \qquad h^{1,0} = 0\,, \qquad h^{2,0} = 0\,, \qquad h^{3,0} = 0\,.$$

For $C_{\tilde{p}} \cdot C_{\tilde{p}} = -2$, the points $p_j, 2 \leq j \leq k$ lie in a two dimensional face. Using the methods described above, we find

$$\chi(D_{p_j}, \mathcal{O}(D_{p_j})) = 0 = -K_{B_2} \cdot C_{\tilde{p}}\,,$$

and

$$h^{0,0} = 1\,, \qquad h^{1,0} = 1\,, \qquad h^{2,0} = 0\,, \qquad h^{3,0} = 0\,.$$

For the points p_1 and p_{k+1}, we get

$$\chi(D_{p_1}, \mathcal{O}(D_{p_1})) = \chi(D_{p_{k+1}}, \mathcal{O}(D_{p_{k+1}})) = 1\,,$$

and

$$h^{0,0} = 1\,, \qquad h^{1,0} = 0\,, \qquad h^{2,0} = 0\,, \qquad h^{3,0} = 0\,.$$

We thus see that the toric method yields results which are in general agreement with formulas (5.3), (5.4), and (5.12) with $g(C_{\tilde{p}}) = 0$, since all the (toric) divisors $C_{\tilde{p}}$ in the toric variety B_2 are rational.

Another way to compute the arithmetic genus of the horizontal divisors is to use $\chi(D, \mathcal{O}(D)) = 1/2 K_{B_3} \cdot \tilde{D}^2$, which expresses the arithmetic genus of the vertical divisor D in the fourfold in terms of its image $\tilde{D}$ in the base B_3, and compute the right hand side from the fan of B_3. As an illustrative example, we explicitly compute the arithmetic genera of the divisors in the Calabi–Yau fourfold fibered over the base shown in fig. 5*b*. Now, each ray in the fan of the base corresponds to a divisor, and the canonical class $K = -\Sigma_i D_i$. Furthermore, three (distinct) divisors intersect if and only if they form a cone in the fan. Thus, for instance, $D_1 \cdot D_2 \cdot D_5 = 1$, but $D_5 \cdot D_4 \cdot D_6 = 0$. In addition, there are three linear relations among the divisors imposed by the structure of the lattice. These relations are:

$$\begin{aligned} D_2 &= D_4 \\ D_1 &= nD_2 + D_7 + D_3 + D_8 + D_9 \\ D_5 + D_7 &= D_8 + 2D_9 + D_6 \,. \end{aligned} \tag{5.16}$$

For the fan of fig. 5*a*, the corresponding relations are identical except for the absence of the terms involving D_7, D_8, D_9, since these divisors are absent. In this case, we recognize D_5, D_6 as giving the fan of the P^1 fiber, and $D_1, \ldots, D_4$ as giving the fan of F_n, with $D_2 = D_4 = f$, $D_1 = C_\infty$, and $D_3 = C_0$. (Strictly speaking, $D_1 = P^*(C_\infty)$, and so on). The fan of fig. 5*b* then describes the variety obtained by blowing up $F_n \times P^1$ three times over C_0, the zero section of F_n.

After a simple computation, we obtain the following results:

$$\begin{aligned} K \cdot D_1^2 &= -2n = -2C_\infty^2, \\ K \cdot D_2^2 &= K \cdot D_4^2 = 0 = -2f^2, \\ K \cdot D_7^2 &= K \cdot D_9^2 = 2, \\ K \cdot D_3^2 &= K \cdot D_8^2 = 4 - 2n = -2K_{F_n} \cdot C_0 \,. \end{aligned} \tag{5.17}$$

The above results again agree with equations (5.1), (5.4) and (5.12).

5.2 Physical implications

The above results have interesting physical implications. Note first that the behavior of the outer divisors in the chain is different from that of the inner divisors. In turn, all inner divisors have the same number of zero modes. This is actually the expected behavior since the outer and inner divisors correspond to two different types of open membranes as explained above.

Although we have derived a necessary and sufficient formula for the generation of a nonperturbative superpotential very little is known about its explicit dependence on the complex structure moduli. A notable exception is the case treated in [49]. Considering an exceptional divisor D_i in the chain, the size of the P^1 fiber ϕ_i combines with a Kaluza-Klein mode obtained by reducing the ten dimensional four-form $C^{(4)}$

on the harmonic form dual to E_i in B_3, resulting in a chiral multiplet Φ_i. The general expression of the corresponding superpotential term is of the form

$$V \sim e^{-\Phi_i} f(\ldots)\,, \tag{5.18}$$

where $f(\ldots)$ is an unknown holomorphic function depending on the complex structure moduli of X as well as the Φ_j and on the positions of the background three-branes [8, 50]. Furthermore, taking into account the possible extremal transitions discussed in the previous section the classical moduli space has a very complicated structure. In this case, there could be contributions to the superpotential of perturbative nature as explained in the simpler context of $N = 2$ $d = 3$ gauge theories in [51]. A discussion of these phenomena in a particular geometric situation has appeared in [52]. Therefore the analysis of this section is a small step towards a complete understanding of this complicated moduli space.

Another interesting aspect is related to the fact that when the number of zero modes forbids the generation of a superpotential, there could be other associated nonperturbative effects. This is well known in the context of $N = 1$ field theories with $N_f = N_c$ when such an effect results in a quantum deformation of the classical moduli space [38]. As also explained in [53] such a phenomenon can be produced by a divisor with $\chi = 0$. In the present situation this is obviously the case when a horizontal heterotic fivebrane wraps an elliptic curve in $C \subset B_2$, according to (5.4). This suggests that in this case the four dimensional fivebrane–instanton transition can be "smoothed" by nonperturbative effects. However, extra care is needed since if $g(C) = 1$, we also have $h^{0,1} = 1$, hence a cancelation could in principle take place.

Acknowledgments

It is a pleasure to thank M. Berkooz, P. Candelas, G. Curio, K. Dasgupta, A. Grassi, J. Gomis, K. Intriligator, B. Ovrut, N. Seiberg, S. Sethi, A. Uranga, E. Witten and Z. Yin for useful discussions. The work of D-E. D. is supported in part by DOE grant DE-FG02-90ER40542. The work of G.R. is supported in part by NSF grant Math/Phys DMS-9627351.

References

[1] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F Theory*, *Comm. Math. Phys.* **187** (1997) 679 [hep-th/9701162].

[2] B. Andreas and G. Curio, *Three-branes and five-branes in N=1 dual string pairs*, *Phys. Lett.* **B 417** (1998) 41 [hep-th/9706093].

[3] S. Sethi, C. Vafa and E. Witten, *Constraints on low-dimensional string compactifications*, *Nucl. Phys.* **B 480** (1996) 213 [hep-th/9606122].

[4] G. Rajesh, *Toric geometry and F-theory/Heterotic Duality in Four Dimensions*, *J. High Energy Phys.* **12** (1998) 018 [hep-th/9811240].

[5] D.R. Morrison and C. Vafa, *Compactifications of f-theory on Calabi–Yau threefolds – I*, *Nucl. Phys.* **B 473** (1996) 74 [hep-th/9602114]; *Compactifications of F-Theory on Calabi–Yau Threefolds – II*, *Nucl. Phys.* **B 476** (1996) 437 [hep-th/9603161].

[6] P.S. Aspinwall, *K3 surfaces in string duality*, TASI(96), hep-th/9611137.

[7] P.S. Aspinwall and D. R. Morrison, *Point-like instantons on K3 orbifolds*, *Nucl. Phys.* **B 503** (1997) 533 [hep-th/9705104].

[8] E. Witten, *Solutions of Four-Dimensional Field Theory via M Theory*, *Nucl. Phys.* **B 500** (1997) 3 [hep-th/9703166].

[9] A. Grassi, *Divisors on elliptic Calabi-Yau 4-folds and the superpotential in F-Theory, I*, alg-geom/9704008.

[10] M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, *On four-dimensional compactifications of F-Theory*, *Nucl. Phys.* **B 505** (1997) 165 [hep-th/9701165].

[11] R. Friedman, J.W. Morgan and E. Witten, *Vector bundles over elliptic fibrations*, alg-geom/9709029.

[12] G. Curio and R.Y. Donagi, *Moduli in N=1 heterotic/F-theory duality*, *Nucl. Phys.* **B 518** (1998) 603 [hep-th/9801057].

[13] P. Berglund and P. Mayr, *Heterotic string/F-theory duality from mirror symmetry*, hep-th/9811217.

[14] P.S. Aspinwall, *Aspects of the hypermultiplet moduli space in string duality*, *J. High Energy Phys.* **04** (1998) 019 [hep-th/9802194].

[15] S. Katz and C. Vafa, *Geometric engineering of N=1 quantum field theories*, *Nucl. Phys.* **B 497** (1997) 196 [hep-th/9611090].

[16] C. Vafa, *On N=1 Yang-Mills in four dimensions*, *Adv. Theor. Math. Phys.* **2** (1998) 497 [hep-th/9801139].

[17] C.H. Clemens and P.A. Griffiths, *The intermediate jacobian of the cubic threefold*, Ann. of Math. (2) **95** (1972) 281.

[18] A.N. Tyurin, *Five lectures on three-dimensional varieties*, Russian. Math. Surveys. (5) **27** (1972) 1.

[19] A. Lukas, B.A. Ovrut and D. Waldram, *Non-standard embedding and five-branes in heterotic M-Theory*, *Phys. Rev.* **D 59** (1999) 106005 [hep-th/9808101].

[20] R. Donagi, A. Lukas, B.A. Ovrut and D. Waldram, *Non-perturbative vacua and particle physics in M-Theory*, *J. High Energy Phys.* **05** (1999) 018 [hep-th/9811168].

[21] E. Sharpe, *Extremal transitions in heterotic string theory*, *Nucl. Phys.* **B 513** (1998) 175 [hep-th/9705210].

[22] A. Hanany and I. R. Klebanov, *On tensionless strings in* 3+1 *dimensions*, *Nucl. Phys.* **B 482** (1996) 105 [hep-th/9606136].

[23] S. Kachru, Y. Oz and Z. Yin, *Matrix description of intersecting M5 branes*, *J. High Energy Phys.* **11** (1998) 004 [hep-th/9803050].

[24] N. Seiberg, *IR dynamics on branes and space-time geometry*, *Phys. Lett.* **B 384** (1996) 81 [hep-th/9606017].

[25] I. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[26] D. Morrison and M.R. Plesser, *Non-spherical horizons, I*, hep-th/9810201.

[27] K. Oh and R. Tatar, *Three dimensional SCFT from M2 branes at conifold singularities*, *J. High Energy Phys.* **02** (1999) 025 [hep-th/9810244].

[28] A. Uranga, *Brane configurations for branes at conifolds*, *J. High Energy Phys.* **01** (1999) 022 [hep-th/9811004].

[29] M. Douglas and G. Moore, *D-branes, quivers, and ALE instantons*, hep-th/9603167.

[30] R. de Mello Koch, K. Oh and R. Tatar, *Moduli space for conifolds as intersection of orthogonal D6 branes*, hep-th/9812097.

[31] A. Giveon and D. Kutasov, *Brane dynamics and gauge theory*, hep-th/9802067.

[32] S. Gubser and I. Klebanov, *Baryons and domain walls in an $N = 1$ superconformal gauge theory*, *Phys. Rev.* **D 58** (1998) 125025, [hep-th/9808075].

[33] A. Strominger, *Massless black holes and conifolds in string theory*, *Nucl. Phys.* **B 451** (1995) 96 [hep-th/9504090].

[34] B. Greene, D.R. Morrison and A. Strominger, *Black hole condensation and the unification of string vacua*, *Nucl. Phys.* **B 451** (1995) 109 [hep-th/9504145].

[35] B. Greene, D.R. Morrison and C. Vafa, *A geometric realization of confinement*, *Nucl. Phys.* **B 481** (1996) 513 [hep-th/9608039].

[36] M. Bershadsky, V. Sadov and C. Vafa, *D-strings on D-manifolds*, *Nucl. Phys.* **B 463** (1996) 398 [hep-th/9510225].

[37] K. Hori, H. Ooguri, C. Vafa, *Non-abelian conifold transitions and N=4 dualities in three dimensions*, *Nucl. Phys.* **B 504** (1997) 147 [hep-th/9705220].

[38] N. Seiberg, *Exact results on the space of vacua of four dimensional SUSY gauge theories*, *Phys. Rev.* **D 49** (1994) 6857, [hep-th/9402044].

[39] K. Intriligator, *RG fixed points in six dimensions via branes at orbifold singularities*, *Nucl. Phys.* **B 496** (1997) 177 [hep-th/9702038].

[40] J.D. Blum and K. Intriligator, *New phases of string theory and 6d RG fixed points via branes at orbifold singularities*, *Nucl. Phys.* **B 506** (1997) 199 [hep-th/9705044].

[41] A. M. Uranga, private communication.

[42] E. Witten, private communication.

[43] A. Grassi, to appear.

[44] P. Mayr, *Mirror symmetry, N=1 superpotentials and tensionless strings on Calabi-Yau four-folds*, *Nucl. Phys.* **B 494** (1997) 489 [hep-th/9610162].

[45] M. Kreuzer and H. Skarke, *Calabi–Yau 4-folds and Toric fibrations*, *J. Geom. Phys.* **26** (1998) 272 [hep-th/9701175].

[46] H. Skarke, *String dualities and toric geometry: an introduction*, hep-th/9806059.

[47] A. Klemm, B. Lian, S-S. Roan and S-T. Yau, *Calabi-Yau fourfolds for M- and F-Theory compactifications*, *Nucl. Phys.* **B 518** (1998) 515 [hep-th/9701023].

[48] A. Avram, M. Kreuzer, M. Mandelberg and H. Skarke, *Searching for K3 fibrations*, *Nucl. Phys.* **B 494** (1997) 567 [hep-th/9610154].

[49] R. Donagi, A. Grassi and E. Witten, *A non-perturbative superpotential with E_8 symmetry*, *Mod. Phys. Lett.* **A 11** (1996) 2199 [hep-th/9607091].

[50] O. Ganor, *A note on zeroes of superpotentials in F-Theory*, *Nucl. Phys.* **B 499** (1997) 55 [hep-th/9612077].

[51] O. Aharony, *Aspects of N=2 supersymmetric gauge theories in three dimensions*, *Nucl. Phys.* **B 499** (1997) 67 [hep-th/9703110].

[52] D.-E. Diaconescu and S. Gukov, *Three dimensional N=2 gauge theories and degenerations of Calabi-Yau four-folds*, *Nucl. Phys.* **B 535** (1998) 171 [hep-th/9804059].

[53] M. Bershadsky, A. Johansen, T. Pantev, V. Sadov and C. Vafa, *F-theory, geometric engineering and N=1 dualities*, *Nucl. Phys.* **B 505** (1997) 153 [hep-th/9612052].